Matrix Formulation of the Subsidiary Conditions of the Second Kind for the Pauli-Fierz Wave Equation

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It is demonstrated, using the Pauli-Fierz wave equation as an example, that the subsidiary conditions of the second kind for a first order wave equation can be expressed in matrix language.

In a series of letters [1, 2, 3, 4] we saw how the subsidiary conditions of the second kind (which are important in studing the propagation of a wave equation in an external field) can be verified by employing spinor calculus. This method has the disadvantage that it is rather complicated. Thus in this paper we show how the same problem can be formulated in matrix language. We use as a prototype the Pauli-Fierz wave equation [5] and show that its subsidiary conditions of the second kind can be expressed in matrix form. In a subsequent paper we shall derive general conditions which have to be satisfied such that the subsidiary conditions of the second kind for any first order wave equation be derivable.

For the moment let us consider the Pauli-Fierz wave equation given in [1, 2]. As it was shown there, this wave equation accepts the following subsidiary conditions of the second kind:

$$\begin{aligned}6\varkappa^2 d^{\dot{1}} &+ (f_{10}+f_{13}+i f_{02}+i f_{32})\, a^{\dot{1}}_{12} + 2(f_{03}+i f_{12})\, a^{\dot{1}}_{12}\\ &+ (f_{01}+i f_{02}+f_{13}+i f_{23})\, \alpha^{\dot{1}}_{22} + 2(f_{03}-i f_{12})\, d^{\dot{1}}\\ &+ 2(f_{01}+i f_{20}-f_{13}-i f_{23})\, d^{\dot{2}} = 0,\end{aligned} \quad (1)$$

$$\begin{aligned}6\varkappa^2 d^{\dot{2}} &+ (-f_{01}-i f_{20}+f_{13}+i f_{32})\, a^{\dot{2}}_{11}\\ &+ 2(-f_{30}+i f_{12})\, a^{\dot{2}}_{12} + (-f_{10}+f_{13}-i f_{20}+i f_{23})\, a^{\dot{2}}_{22}\\ &+ 2(f_{30}-i f_{21})\, d^{\dot{2}} + 2(f_{01}-f_{31}+i f_{20}-i f_{23})\, d^{\dot{1}} = 0,\end{aligned} \quad (2)$$

$$\begin{aligned}6\varkappa^2 c_1 &+ (-f_{01}+f_{31}-i f_{20}+i f_{23})\, b^{\dot{1}\dot{1}}_1 + 2(i f_{21}-f_{30})\, b^{\dot{1}\dot{2}}_1\\ &+ (-f_{10}-i f_{20}+f_{31}+i f_{32})\, b^{\dot{2}\dot{2}}_1 + 2(f_{30}-i f_{12})\, c_1\\ &+ 2(f_{10}+i f_{20}-f_{13}-i f_{23})\, c_2 = 0,\end{aligned} \quad (3)$$

$$\begin{aligned}6\varkappa^2 c_2 &+ (-f_{01}+f_{31}-i f_{20}+i f_{23})\, b^{\dot{1}\dot{1}}_2\\ &+ 2(-f_{30}+i f_{21})\, b^{\dot{1}\dot{2}}_2 + (-f_{10}+f_{31}-i f_{20}+i f_{32})\, b^{\dot{2}\dot{2}}_2\\ &+ 2(f_{10}-f_{31}+i f_{02}-i f_{23})\, c_1 + 2(-i f_{21}+f_{03})\, c_2 = 0.\end{aligned} \quad (4)$$

Let us for instance consider the derivation of the subsidiary condition (1). In deriving this subsidiary condition the following steps were involved (see [2]). First construct

$$\{\pi^{\dot{1}}_{1}(e q_9) + \pi^{\dot{1}}_{2}(e q_{10}) + \pi^{\dot{2}}_{1}(e q_{12}) + \pi^{\dot{2}}_{2}(e q_{13})\} - \{3\pi^{1\dot{1}}(e q_{15}) - 3\pi^{2\dot{1}}(e q_{16})\} = 0 \quad (5)$$

and substitute into it $6\varkappa^6 (e q_7)$ having replaced $\pi_{a\dot{\varrho}}$ to produce terms involving $[\pi_k, \pi_l] = i e F_{kl} = f_{kl}$.

The above steps can be put into matrix language as follows:

$$\mathbb{S}^{(\mathrm{I})} \{\mathbb{A}^{(\mathrm{I})}_0 \pi_0 + \mathbb{A}^{(\mathrm{I})}_1 \pi_1 + \mathbb{A}^{(\mathrm{I})}_2 p_2 + \mathbb{A}^{(\mathrm{I})}_3 \pi_3 + \mathbb{A}^{(\mathrm{I})}_4 \varkappa\} \cdot \{-\mathbb{L}_0 \pi_0 + \mathbb{L}_1 \pi_1 + \mathbb{L}_2 \pi_2 + \mathbb{L}_3 \pi_3 + \mathbb{1}\varkappa\}\, \psi = 0, \quad (6)$$

where $\mathbb{S}^{(\mathrm{I})}$ is an 1×16 matrix and $\mathbb{A}^{(\mathrm{I})}_0$, $\mathbb{A}^{(\mathrm{I})}_1$, $\mathbb{A}^{(\mathrm{I})}_2$, $\mathbb{A}^{(\mathrm{I})}_3$, $\mathbb{A}^{(\mathrm{I})}_4$ are 16×16 matrices. The index (I) makes references to the first subsidiary condition of the second kind. The matrices $\mathbb{S}^{(\mathrm{I})}$, $\mathbb{A}^{(\mathrm{I})}_j$, $j = 0, 1, 2, 3, 4$, with respect to the spinor basis

$$\{a^{\dot{1}}_{11}, a^{\dot{1}}_{12}, a^{\dot{1}}_{22}, a^{\dot{2}}_{11}, a^{\dot{2}}_{12}, a^{\dot{2}}_{22}, d^{\dot{1}}, d^{\dot{2}}, b^{\dot{1}\dot{1}}_1, b^{\dot{1}\dot{2}}_1, b^{\dot{2}\dot{2}}_1, b^{\dot{1}\dot{1}}_2, b^{\dot{1}\dot{2}}_2, b^{\dot{2}\dot{2}}_2, c_1, c_2\} \quad (7)$$

are

$$\mathbb{S}^{(\mathrm{I})} = [0, 0, 0, 0, 0, 0, 6, 0, 1, 1, 0, 1, 1, 0, -3, -3], \quad (8)$$

$$\mathbb{A}^{(\mathrm{I})}_0 = \mathrm{diag}[0,0,0,0,0,0,0,0,-1,0,1,0,0,-1,0], \quad (9)$$

$$\mathbb{A}^{(\mathrm{I})}_1 = \mathrm{diag}[0,0,0,0,0,0,0,0,1,0,0,0,-1,0,0,-1], \quad (10)$$

$$\mathbb{A}^{(\mathrm{I})}_2 = \mathrm{diag}[0,0,0,0,0,0,0,0,-i,0,0,0,-i,0,0,-i], \quad (11)$$

$$\mathbb{A}^{(\mathrm{I})}_3 = \mathrm{diag}[0,0,0,0,0,0,0,0,0,-1,0,-1,0,0,-1,0], \quad (12)$$

$$\mathbb{A}^{(\mathrm{I})}_4 = \mathrm{diag}[0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0]. \quad (13)$$

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The matrices for the other three subsidiary conditions are, respectively,

$$\mathbb{S}^{(\mathrm{II})} = [0,0,0,0,0,0,0,6,0,1,1,0,1,1,-3,-3], \quad (14)$$

$$\mathbb{A}_0^{(\mathrm{II})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,-1], \quad (15)$$

$$\mathbb{A}_1^{(\mathrm{II})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,1,0,0,0,-1,-1,0], \quad (16)$$

$$\mathbb{A}_2^{(\mathrm{II})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,-i,0,0,0,-i,i,0], \quad (17)$$

$$\mathbb{A}_3^{(\mathrm{II})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,0,-1,0,-1,0,0,1], \quad (18)$$

$$\mathbb{A}_4^{(\mathrm{II})} = \mathrm{diag}[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0], \quad (19)$$

$$\mathbb{S}^{(\mathrm{III})} = [1,1,0,1,1,0,-3,-3,0,0,0,0,0,0,6,0], \quad (20)$$

$$\mathbb{A}_0^{(\mathrm{III})} = \mathrm{diag}[0,1,0,-1,0,0,-1,0,0,0,0,0,0,0,0,0], \quad (21)$$

$$\mathbb{A}_1^{(\mathrm{III})} = \mathrm{diag}[1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0,, \quad (22)$$

$$\mathbb{A}_2^{(\mathrm{III})} = \mathrm{diag}[-i,0,0,0,-i,0,0,i,0,0,0,0,0,0,0,0], \quad (23)$$

$$\mathbb{A}_3^{(\mathrm{III})} = \mathrm{diag}[0,-1,0,-1,0,0,1,0,0,0,0,0,0,0,0,0], \quad (24)$$

$$\mathbb{A}_4^{(\mathrm{III})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0], \quad (25)$$

$$\mathbb{S}^{(\mathrm{IV})} = [0,1,1,0,1,1,-3,-3,0,0,0,0,0,0,0,6], \quad (26)$$

$$\mathbb{A}_0^{(\mathrm{IV})} = \mathrm{diag}[0,0,1,0,-1,0,0,-1,0,0,0,0,0,0,0,0], \quad (27)$$

$$\mathbb{A}_1^{(\mathrm{IV})} = \mathrm{diag}[1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0], \quad (28)$$

$$\mathbb{A}_2^{(\mathrm{IV})} = \mathrm{diag}[-i,0,0,0,-i,0,0,i,0,0,0,0,0,0,0,0], \quad (29)$$

$$\mathbb{A}_3^{(\mathrm{IV})} = \mathrm{diag}[0,-1,0,-1,0,0,1,0,0,0,0,0,0,0,0,0], \quad (30)$$

$$\mathbb{A}_4^{(\mathrm{IV})} = \mathrm{diag}[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1]. \quad (31)$$

If $\mathbb{S}$ is considered as a 4×20 matrix and $\mathbb{A}_0$, $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$ are considered as 20×16 matrices then all four subsidiary conditions can be accomodated into one matrix formula, namely

$$\mathbb{S}\{\mathbb{A}_0 \pi_0 + \mathbb{A}_1 \pi_1 + \mathbb{A}_2 \pi_2 + \mathbb{A}_3 \pi_3 + \mathbb{A}_4 \varkappa\} \cdot \{-\mathbb{L}_0 \pi_0 + \mathbb{L}_1 \pi_1 + \mathbb{L}_2 \pi_2 + \mathbb{L}_3 \pi_3 + \mathbb{1} \varkappa\} \psi = 0.$$

Thus we have demonstrated with this example that the subsidiary conditions of the second kind of a first order wave equation can be derived using matrix calculus. The question which arises is what conditions must be satisfied such that the subsidiary conditions of the second kind for any first order wave equation be derivable. We shall deal with this question in a subsequent paper.

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